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Field-theoretic version of a two-dimensional Coulomb gas with repulsive cores

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The statistical mechanics of a two-dimensional gas of charged particles *with repulsive short-distance cores*, in addition to their electrostatic Coulomb interactions, is recast into a sine-Gordon-like field theory. The Kosterlitz-Thouless renormalization-group flow equations are readily obtained. The theory may be expressed in terms of fermionic variables.

The two-dimensional Coulomb gas, consisting of positive and negative charges, has attracted considerable attention.[1–6] The primary reason is its equivalence to the statistical mechanics of vortices that appear in the planar X-Y model. The critical properties of this gas were presented by Kosterlitz and Thouless[1] and by Kosterlitz[2] using very physical renormalization-group arguments. With different techniques, José *et al.*[3] obtained the same flow equations. In all these treatments the charged particles are assumed to have some sort of repulsive hard core preventing arbitrarily close approaches and thus avoiding the singular part of the Coulomb potential.

Ignoring the last point, the Coulomb gas is equivalent to the sine-Gordon field theory specified by the Lagrangian[7,8]

$$L_{\mathrm{SG}}=\tfrac{1}{2}(\nabla\varphi)^2-K\cos(4\pi\beta q^2)^{1/2}\varphi \ . \tag{1}$$

$\beta=1/k_BT$, q is the magnitude of the charge on each particle, and K is related to the fugacity. Coleman[9] showed that this theory does not have a ground state for $\beta q^2\geqslant 2$. $\beta q^2=2$ is precisely the fixed point of Refs. 1–3.

The inclusion of some sort of short-distance cutoff is crucial; it is the only relevant distance or mass scale present in this problem. It was realized by Samuel,[8] Wiegmann,[5] and by Amit, Goldschmidt, and Grinstein[6] that the root of the problem, for $\beta q^2>2$, was that the infinite set of Feynman diagrams for each order in K develop upon summation, a singularity. This singularity can be handled by a wave-function renormalization. This renormalization is introduced in addition to the normal ordering discussed by Coleman. These two ingredients of the renormalization program are sufficient for the purpose of obtaining the Kosterlitz-Thouless renormalization-group flow equations.[1,2] However, the relation between a wave-function renormalization and a repulsive hard-core potential of the Coulomb gas is not direct.

We shall present a regularization scheme based not on hard-core repulsion, but on a softer Yukawa repulsion between the Coulomb gas particles. This formulation can be readily transcribed into a local-field theory involving two Bose fields, one the original sine-Gordon field φ of Eq. (1) and an auxiliary massive field χ. Several interesting and useful properties of this formalism are the following.

(i) It is free of all the diseases of the sine-Gordon theory at large β.

(ii) It yields, with a minimum of effort, the Kosterlitz-Thouless flow equations. Higher-order terms in these equations can also be obtained.[5,6]

(iii) It has a fermionic transcription.

(iv) It can serve as a basis for approximate calculations in both phases of the system. (We hope to return to this point in a future work.) A major disadvantage is that the Lagrangian in this formulation is not real.

The Hamiltonian we shall study is

$$H=\sum_{i>j}[-q_iq_j\ln\mu^2|r_i-r_j|^2+2q^2K_0(m|r_i-r_j|)]+\sum_i\epsilon_0 \ . \tag{2}$$

The first term is the usual Coulomb Hamiltonian; the second term is the aforementioned Yukawa repulsion, chosen in such a way as to completely cancel the singularities at $r_i=r_j$. The last term is the self-energy or chemical potential of the gas particles. The charges are $q_i=\pm q$. In the limit $m\to\infty$ the Yukawa repulsion becomes inoperative. As we shall see this analytic repulsion, for $r<1/m$, has many advantages.

The partition function corresponding to this Hamiltonian is

$$Z=\sum_{n,m}\frac{(K/2)^{n+m}}{n!m!}\int d^2x_1\cdots d^2x_nd^2y_1\cdots d^2y_m\,\exp\left(2\beta\sum[q_iq_j\ln\mu|r_i-r_j|-q^2K_0(m|r_i-r_j|)]\right) \ , \tag{3}$$

with $r_i = x_i$ for $q_i = +q$, $r_i = y_i$ for $q_i = -q$, and $\frac{1}{2}K = \exp(-\beta\epsilon_0)$. Due to the repulsion, Z converges term by term. In terms of the aforementioned fields Z is equivalent to the following path integral[8]

$$Z = \int [d\varphi][d\chi] \exp\left\{-\int d^2r [\tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}(\nabla\chi)^2 + \tfrac{1}{2}m^2\chi^2 - K e^{i(4\pi\beta q^2)^{1/2}\chi} \cos(4\pi\beta q^2)^{1/2}\varphi]\right\} . \tag{4}$$

For $\chi = 0$ this is just the vacuum-to-vacuum transition amplitude of the sine-Gordon theory. Note the Lagrangian of Eq. (4) is not real; this prevents us from performing variational calculations. We emphasize that it is finite, order by order in K, for all values of βq^2.

The Kosterlitz-Thouless equations can be obtained after an approximate integration over the field χ. The resulting theory, involving only the field φ is based on a real Lagrangian and is finite for some range of $\beta q^2 > 2$. The basis for the approximation is the recognition of the fact that $\chi \sim 1/m$. We rewrite Eq. (4) as

$$\begin{aligned} Z &= \left\langle \exp K \int d^2r e^{i(4\pi\beta q^2)^{1/2}\chi} \cos(4\pi\beta q^2)^{1/2}\varphi \right\rangle_{\chi,\varphi} \\ &= \left\langle \left[\exp\left(K\int d^2r \cos(4\pi\beta q^2)^{1/2}\varphi\right)\right]\left[\exp\left(K\int d^2r (e^{i(4\pi\beta q^2)^{1/2}\chi} - 1)\cos(4\pi\beta q^2)^{1/2}\varphi\right)\right]\right\rangle_{\chi,\varphi} . \end{aligned} \tag{5}$$

The subscripts φ and χ indicate that the average is to be taken with respect to the massless field φ and the massive field χ. Normal ordering with respect to a mass m is assumed for both fields. (Normal ordering with respect to some other mass can be reabsorbed into a redefinition of K.) Treating $\exp[i(4\pi\beta q^2)^{1/2}\chi] - 1$ as a small term we may perform the averaging over the field χ and find

$$\begin{aligned} Z \approx \Bigg\langle &\left[\exp\left(K\int d^2r \cos(4\pi\beta q^2)^{1/2}\varphi\right)\right] \\ &\times\left[\exp\left(\frac{-K^2}{2}\int d^2r_1 d^2r_2 \{1 - \exp[-2\beta q^2 K_0(m|r_1 - r_2|)]\}\cos(4\pi\beta q^2)^{1/2}\varphi(r_1)\cos(4\pi\beta q^2)^{1/2}\varphi(r_2)\right)\right]\Bigg\rangle_{\varphi} . \end{aligned} \tag{6}$$

The integration in the second term is restricted to $|r_1 - r_2| < 1/m$. This approximation forces us to introduce, temporarily, an artificial cutoff Λ such that $|r_1 - r_2| > 1/\Lambda$. This dependence on Λ disappears when we take the first term in Eq. (6) to second order in K (we checked this result by explicit calculations). A sufficiently accurate result is obtained by replacing the second term in Eq. (6) by

$$\frac{K^2}{2}\int d^2r \int_{1/\Lambda}^{1/m} d^2\epsilon N_m \cos(4\pi\beta q^2)^{1/2}\varphi(r + \tfrac{1}{2}\epsilon) N_m \cos(4\pi\beta q^2)^{1/2}\varphi(r - \tfrac{1}{2}\epsilon) . \tag{7}$$

Normal ordering with respect to m is explicitly indicated. In order to isolate the singular part of Eq. (7) we bring the two terms in the integrand to a common ordering. Aside from terms finite in the $\epsilon \to 0$ limit we obtain the singular part of Eq. (7)

$$\begin{aligned} &\frac{K^2}{2}\int d^2r \int_{1/\Lambda}^{1/m} d^2\epsilon \tfrac{1}{2}(m^2\epsilon^2)^{-\beta q^2} N_m \cos(4\pi\beta q^2)^{1/2}[\varphi(r + \tfrac{1}{2}\epsilon) - \varphi(r + \tfrac{1}{2}\epsilon)] \\ &\qquad \approx \frac{K^2}{4}\int d^2r \int_{1/\Lambda}^{1/m} d^2\epsilon (m^2\epsilon^2)^{-\beta q^2}[1 - 2\pi\beta q^2(\epsilon\nabla\varphi)^2] \\ &\qquad \approx \int d^2r \frac{\pi}{4}\frac{K^2}{(m^2)^{\beta q^2}}\left\{(m^2)^{\beta q^2 - 1}\frac{\left(\frac{\Lambda^2}{m^2}\right)^{\beta q^2 - 1} - 1}{\beta q^2 - 1} - \pi\beta q^2 (m^2)^{\beta q^2 - 2}\frac{\left(\frac{\Lambda^2}{m^2}\right)^{\beta q^2 - 2} - 1}{\beta q^2 - 2}(\nabla\varphi)^2\right\} . \end{aligned} \tag{8}$$

The first term is an additive constant that contributes only to the free energy.[6] The second one is the wave-function renormalization. Expanding this term around $\beta q^2 = 2$, we obtain an effective Lagrangian involving only the field φ

$$\begin{aligned} L = &\tfrac{1}{2}\left\{1 - \pi^2\beta q^2\frac{K^2}{m^4}\ln\frac{\Lambda}{m}\right\}(\nabla\varphi)^2 \\ &- K N_m \cos(4\pi\beta q^2)^{1/2}\varphi . \end{aligned} \tag{9}$$

The Kosterlitz-Thouless equations are obtained by renormal ordering the cosine term from the mass m to some arbitrary fixed mass μ, and rescaling the first term to bring it to standard form. Using[9]

$$N_m \cos(4\pi\beta q^2)^{1/2}\varphi = \left(\frac{\mu^2}{m^2}\right)^{\beta q^2/2} N_\mu \cos(4\pi\beta q^2)^{1/2}\varphi , \tag{10}$$

the Lagrangian becomes

$$L=\tfrac{1}{2}(\nabla\varphi)^2-\mu^2 y\left(\frac{\mu^2}{m^2}\right)^{\beta q^2/2-1} N_\mu \cos[4\pi(\beta q^2)']^{1/2}\varphi \;, \tag{11}$$

with

$$y=K/m^2;\quad (\beta q^2)'=\beta q^2[1+\pi^2\beta q^2 y^2 \ln(\Lambda/m)] \;. \tag{12}$$

Requiring that $y(\mu^2/m^2)^{\beta q^2/2-1}$ and $(\beta q^2)'$ be unchanged under a scale change $m \to m+\delta m$, we obtain the equations

$$dy=y(\beta q^2-2)(dm/m) \;,$$

$$d(\beta q^2)=\pi^2(\beta q^2)^2 y^2(dm/m) \;.$$

The first equation is unaffected by a change of scale, since it can be put in a form that involves only the scale-invariant differentials dy^2/y^2 and dm/m. Taking advantage of this fact, in the vicinity of the fixed point the equations may be put in the standard Kosterlitz-Thouless form

$$\tau\frac{dy}{d\tau}=-xy \;,\quad \tau\frac{dx}{d\tau}=-y^2 \;, \tag{13}$$

where $x=\beta q^2-2$ and $\tau=1/m$.

Equation (4) provides the local theory equivalent to a Coulomb gas with a repulsive core. For some purposes it may prove useful to study its fermionized version.[9-11] This involves two Fermi fields, ψ corresponding to φ, η corresponding to χ, and an electromagnetic potential A_μ to take care of the massiveness of the χ field[11]

$$\begin{aligned} L=&\bar\psi i\not\partial\psi+\bar\eta(i\not\partial-e\not A)\eta-\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}\\ &-\frac{g}{2}:\bar\psi\gamma_\mu\psi::\bar\psi\gamma_\mu\psi:-\frac{g}{2}:\bar\eta\gamma_\mu\eta::\bar\eta\gamma_\mu\eta:\\ &-\frac{K}{m^2}:\bar\eta(1+\gamma_5)\eta::\bar\psi\psi: \;, \end{aligned} \tag{14}$$

where

$$g/\pi=1/\beta q^2-1 \;,\quad e=m\sqrt{\pi}/(\beta q^2)^{1/2} \;. \tag{15}$$

The equivalence between Eqs. (14) and (4) holds term by term in K regardless of the magnitude of βq^2.

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